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SCALE AXIS, FRACTALS AND SOME THERMAL PHENOMENA

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Abstract

In this paper, the idea of an objective scale axis enriching Nature with a new dimension is explicated and illustrated on the problem of heat conduction. Physical description of Nature thus has to be formulated on individual scale levels parametrized by points of the scale axis), space and time are intrinsic parameters of each level. A 'global' space-time then becomes a useful construction expressing a possibility of a scale-independent description. Consequently, the cases in which such a description is not possible (e.g. the problem of thermal waves) might lead to a contradiction with the concept of a global space-time, which may manifest itself by a presence of some fractal structures.

Keywords: fractals, heat conduction, scale axis, scale-dependent description, telegrapher equation

Introduction

In physics of this century, the concept of length scale – a characteristic (or smallest) length taken into account in a physical model – has emerged and started to play a very important role. There have been various reasons leading to introduction of this concept: the presence of divergencies in quantum field theories (the 'cutoff' removing these divergencies corresponds to this length [1, 2]), the impossibility of managing microscopic models of systems with many degrees of freedom (e.g. large fluctuations in critical phenomena have been described by employing the concept of scale in the works of Wilson and others, e.g. [3]), the so-called 'size-scaled' effects in macroscopic physics (e.g. the failure of materials and structures [4]), the occurrence of phenomena violating local thermodynamic equilibrium (related to the scale-dependent description outlined in [5]) and many others.

Nevertheless, the 'scale' itself is usually understood as a very suitable but only helpful, 'artificial' concept enabling us to construct reasonable models describing Nature at chosen levels and formulate relations between these descriptions (such as the renormalization group approach). A very important exception is the approach of Nottale (e.g. [6]) who has introduced the concept of scale relativity, being motivated

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by Einstein's theory of relativity. Thereby the very concept of scale has been approached to such fundamental physical concepts as space and time. From a physical point of view, the scale itself is identified with the concept of *resolution* which is modelled by special mathematical objects *fractals*. Another example of grasping the scale as a more fundamental concept is presented in the work of Havel [7], in which the concept of *scale axis* has been introduced enriching Nature with a new dimension – the scale.

In this paper, the idea of scale axis is evolved into a physical approach which uses throughout the concept of scale. As evoked by the word 'axis', various levels of description parametrized by a length parameter l – the scale – are genuinely taken into account. These levels, however, cannot be simply understood as some 'hyperplanes' in a multi-dimensional geometric space. Namely, by introducing a scale axis, the concept of space itself dramatically changes. The reason is that by taking the concept of scale seriously, points on one level (scale) cannot be identified with points on another level (different scale). Namely points of space defined at a scale are 'points with a resolution' and this resolution decreases with increasing scale. As a result, different points can merge into one point during a passage to larger-scale levels. On the other hand, each individual level (a point of the scale axis) should be identified with d-dimensional Euclidean space and, via coordinates of this space, a correspondence between different scale levels could be realized. This, however, contradicts the fact that different points can join together at larger scales. Moreover, it faces the problem that Euclidean space, on a level at scale l, is endowed with an intrinsic, natural length l and the distance between two points is a fixed real number. Thus the distance between corresponding points has to increase when we move 'downward' along the scale axis. As a result, the limit of infinite resolution $l \to 0$ (a zero point of the scale axis) would be a space where the distance between any couple of different points tends to infinity!

In this paper, we will study this problem and show that a solution consists in abandoning a picture of a global space-time in which all physical processes occur. Instead, we propose to understand a 'global' space-time as a useful approximation. By using the problem of heat conduction as an illustration, we will show that the possibility of constructing a suitable 'global' space-time depends on the character of the processes studied. Namely, if the increase of distances during a passage to smaller scales can be compensated by an appropriate change of duration of the process (in an intrinsic time unit of each level), then a reasonable 'global' space-time may be constructed and the equation of the process may be written in a scale-independent form. On the other hand, if there is no such a compensation some manifestation of a singular behaviour in the limit $l \to 0$ might be expected. Because the situation in which the distance between any couple of points tends to infinity may be modelled by using the concept of *fractals* [6], it is expected that fractal structures might play an important role in this singular behaviour. We will show that the problem of heat waves seems to be of this type.

Scale levels of physical description and a 'global' space-time

Following the pioneering work of Havel [7], let us imagine that 'scale' is a new dimension of Nature. Taking this idea seriously we have to assign to each length l (scale) a 'scale level' which will be called the *space at scale l* or simply *l-space*. A passage from l-space to l'-space may be understood as a movement along a *scale axis*, which is nothing else but a 'zoom in' (if $l>l'$) or 'zoom out' if $l<l'$ [7].

A surprising consequence of this idea is its incompatibility with a familiar picture of the world embedded in a global geometric space. Namely, if the world were located in space E then each l-space would be a set $\{(x,l), x\in E\}$. Then, however, the couple of elements (x,l), (x',l) $(x\neq x')$ would represent different points independent of the choice of l, which contradicts the concept of scale level. Namely, if we have two different points A, B on the scale l we can always find a larger scale l' such that these points merge into one because their distance 'disappears' under a resolution corresponding to this scale (Fig. 1).

That is why we will not use a picture of a global geometric space as a 'scene' where all physical processes occur. Instead, we will study any physical process on definite scale levels (l-spaces) in an explicit scale-dependent form, by using intrinsic space-time variables ξ and ν, assigned to each l-space in which the process is studied. The space variable ξ is a d-dimensional vector of real numbers (usualy $d=3$) and it is related to the l-space so that $\|\xi-\xi'\|$ is the distance between the points ξ and ξ' in the natural length unit of this space – the scale l (contrary to Euclidean space where the distance may be an arbitrarily chosen real number!). The time variable (a real number) is chosen with respect to the process under study, so that time $\nu=1$ corresponds to a characteristic time interval of this process.

The key point of the present approach consists in understanding the concept of 'global' space-time as a helpful construct, which may be defined for some physical processes as a useful idealization, or a suitable approximation, enabling us to describe these processes in a scale-independent form, replacing scale-dependent descriptions in a group of individual l-spaces. Such an approximation in a 'global' space-time is constructed by defining 'scale-independent' variables x and t (a 'global' space-time

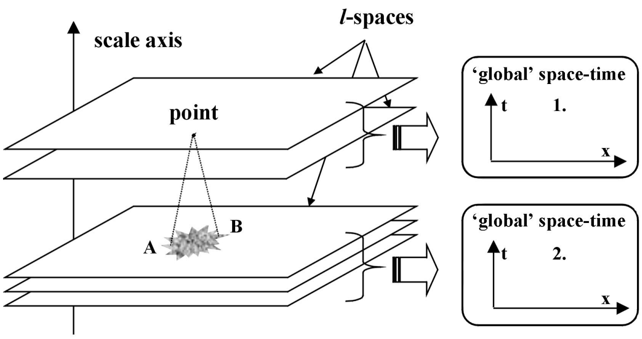

Fig. 1 Various 'global' space-times constructed on l-spaces

variables) as a function of l and intrinsic space-time variables (ξ and ν) of corresponding l-spaces so that equations describing the studied process can be transformed to a scale-independent form. It is worth stressing that various 'global' space-times can be constructed for different groups of l-spaces as is schematically outlined in Fig. 1.

The 'global' variable x is a point of d-dimensional (scale-independent) Euclidean space which has no natural unit of length. Therefore a relation between x and ξ (which is a fixed d-dimensional vector) should be of a form

$$x=l\xi \tag{1}$$

This relation shows the essence of replacing a group of l-spaces by a 'global' space-time. Namely, during a passage across a group of l-spaces (zooming out) a 'spot' created by different points can shade into one point (Fig 1). This, however, contradicts the relation (1), because different points in l-space correspond to different points x, x' in a 'global' space and so, in l'-space with an arbitrarily large l', there are two different points $\xi \equiv x/l'$, $\xi' \equiv x'/l'$ corresponding to x, x' and therefore to those points in l-space. The reason lies in the approximative character of relation (1) for constructing a 'global' space – it cannot be used on a very broad range of l-spaces. The smaller the interval of the scales, the better is the approximation given by a 'global' space (the 'global' space may thus be understood as an analogy of a tangent approximating a smooth curve at a point and its neighbourhood).

The possibility and appropriateness of replacing a group of l-spaces by a 'global' space-time depend on the character of the process studied. Namely, we have to find a relation defining a 'global' time t as a function of scale l and intrinsic time ν of corresponding l-space

$$t=f_l(\nu) \tag{2}$$

The question is whether we are able to find a function $f_l(\nu)$ so that equations describing the physical process in individual l-spaces can be transformed by using (1) and (2) to a scale-independent form. If so, we may describe (approximately) the process in the 'global' space-time defined by relations (1) and (2).

As an example, let us write in l-space the standard equation describing heat conduction without heat sources. In the variables ξ and ν (Δ_ξ is the Laplace operator in these variables) it is of the form

$$\frac{\partial T_l}{\partial \nu} - \kappa(l)\Delta_\xi T_l = 0 \tag{3}$$

It should be emphasized that this is a correct form of the heat conduction equation expressing the fact that the coefficient κ may depend on the scale l on which the temperature T_l is defined (l can be understood as a linear dimension of the active part of a thermometer measuring the temperature).

If we define a 'global' space variable x by the relation (1) and a 'global' time variable t by a transformation

$$t=\nu l^2 \tag{4}$$

Eq. (3) becomes the familiar standard heat conduction equation,

$$\frac{\partial T}{\partial t} - \kappa \Delta T = 0 \tag{5}$$

in which the coefficient κ is supposed to be constant within an interval of scales. In fact, it is just the interval vaguely defined in continuum thermodynamics as ‘regions being small enough to be infinitesimally small from a macroscopic point of view but as large as that may be understood as macroscopic systems’.

Hyperbolic heat conduction equation

Equation (3) cannot, however, describe the problem of heat conduction in all cases. There are some phenomena, such as thermal waves, which cannot be described by a parabolic equation. Therefore a correction leading to a hyperbolic equation has to be made. In *l*-space we obtain the following equation

$$\frac{\partial T_l}{\partial \nu} + \theta(l)\frac{\partial^2 T_l}{\partial \nu^2} - \kappa(l)\Delta_\xi T_l = 0 \tag{6}$$

The coefficient θ(*l*) is a new constant of the description and it has to be experimentally determined. Equation (6) correctly renders the heat conduction, moreover, it does not allow an infinite speed of propagation of a heat pulse.

Now, we will try to construct a ‘global’ space-time, i.e. variables *x*, *t* defined by (1) and (2), transforming Eq. (6) to a scale-independent form. The character of Eq. (6), however, does not allow us to find a function $f_l(\nu)$ so that the constants θ(*l*) and κ(*l*) can be supposed to be scale-independent and the equation would be scale-independent too. For example, the choice (4) leads to the equation

$$\frac{\partial T}{\partial t} + l^2\theta(l)\frac{\partial^2 T}{\partial t^2} - \kappa \Delta T = 0 \tag{7}$$

Supposing that $\kappa(l) \equiv \kappa$ is a scale-independent constant, we can define a new, scale-independent constant, τ

$$\tau \equiv l^2 \theta(l) \tag{8}$$

and obtain the scale-independent, so-called telegrapher equation

$$\frac{\partial T}{\partial t} + \tau \frac{\partial^2 T}{\partial t^2} - \kappa \Delta T = 0 \tag{9}$$

The problem, however, is that heat conduction in individual *l*-spaces becomes rather singular as a result of the fact that, due to the relation (8),

$$\theta(l) \approx l^{-2} \tag{10}$$

Namely, the speed of heat pulses in *l*-space cannot exceed the value

$$c(l)=\sqrt{\frac{\kappa(l)}{\theta(l)}} \tag{11}$$

and, by using (10), we see that in the limit $l\rightarrow 0$ this value vanishes. Hence an (intrinsic) time interval $\Delta\nu$ of any process transmitting information between two different points ξ_1, ξ_2 becomes indefinitely large. In other words, any couple of different points ξ_1, ξ_2 in l-space appear to be more distant if l decreases.

This behaviour seems to have an origin in relation (1). Namely, by fixing a global space at the very beginning, we predetermine the geometry in l-spaces in such a way that any different points x_1, x_2 in the global space become more and more distant in l-spaces if $l\rightarrow 0$, because due to (1) $\|\xi_1-\xi_2\| = l^{-1}\|x_1-x_2\|$ (let us realize that having a global space we can choose a point x in it and find a corresponding point in any l-space). This tendency, however, need not have a physical manifestation if this increase of intrinsic distances can be compensated by an appropriate change of (intrinsic) duration of the process. This is the case of heat conduction described by Eq. (5), in which this compensation has been realized by choosing the transformation (4).

On the other hand, if there is no such a compensation and we work in a global geometric space, we should face the problem that the studied phenomenon would behave at vanishing scales $l\rightarrow 0$ really as if the distance between arbitrary points increased to infinity. This property (the distance between arbitrary points tends to infinity) is a basic attribute of general *fractal* structures, because fractals need not be connected with the concept of self-similarity (see the definition in [6]). In the next section, we will show that a thermodynamic interpretation of Eq. (9) (in a global geometric space) really leads to concepts resembling the description of fractal structures.

The telegrapher equation and fractals

The telegrapher heat conduction Eq. (9) cannot be derived within the framework of local continuum thermodynamics, using an image of infinitesimally small cells in which all thermodynamic quantities may be defined as being in equilibrium (an assumption of *local equilibrium*). An extended description called *extended irreversible thermodynamics* [8] has been developed to generalize thermodynamics beyond the assumption of local equilibrium. The key idea of this approach consists in understanding thermodynamic fluxes (such as heat flux) as new, independent variables. The relation of this generalization to the problem of hyperbolic heat conduction can be easily explained: To obtain the hyperbolic heat equation we need a new constitutive relation connecting the heat flux q with the gradient of temperature. This relation is called the Maxwell–Cataneo law (e.g. [8]) and is of the form

$$q+\tau\frac{\partial q}{\partial t}=-\lambda\nabla T \tag{12}$$

It can be easily demonstrated that this law leads to the hyperbolic Eq. (9). The law (12), however, is rather strange: Namely let us imagine the situation when the temperature field $T(x)$ is constant, i.e. the same at each point. In this situation, the heat flux is expected to be zero. Nevertheless, though the gradient in (12) becomes zero, Eq. (12) allows a solution with a non-zero flux, $q \sim \exp(-t/\tau)$. How can this be explained?

The only way of grasping this solution is to imagine that the flux occurs *inside* individual cells (on which the temperature is defined) while the gradient of temperature expresses a relation between neighbouring cells. In other words, the fluxes in (12) have a different status to those in standard continuum thermodynamics – they are independent variables assigned to each thermodynamic cell.

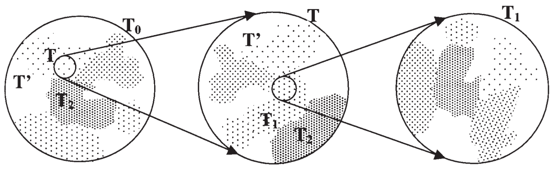

Fig. 2 Fractal distribution of temperature

The presence of independent fluxes within local thermodynamic cells corresponds with the invalidity of the assumption of local equilibrium. Namely, if a cell is in thermal non-equilibrium there are some warmer and colder 'islands' within it. These thermal heterogenities induce some internal fluxes. Now, going to smaller scales, we have to 'see' the same picture: thermal heterogenities and fluxes induced by them (if we were able to find a scale at which the heterogenities vanish, we could define the local equilibrium at this scale). Thus we obtain a general (not self-similar) fractal structure (Fig. 2). It supports the ideas presented in the previous section.

Conclusions

In this paper, the idea of Havel [7] of identifying scale with a new dimension of Nature has been taken into account seriously. The formulation and study of physical processes on individual scale levels, the so-called *l*-spaces, seems to be very natural because we really do that in any experimental work, numerical modelling or computer simulation. Within a framework of a theoretical description, however, we look for a 'general', scale-independent formulation in a (scale-independent!) geometric space or space-time, which might be in conflict with the formulation in *l*-spaces.

Therefore we have outlined the way in which the formulation of a physical problem in *l*-spaces is a starting point and a global (scale-independent) formulation in a geometric space becomes an open problem which has to be solved individually from case to case. Thereby the status of a global formulation (and even of geometric space itself!) dramatically changes. Namely, each such formulation is only a suitable approximation enabling us to work with a group of scale-dependent formulations (e.g. equations parameterized by a scale parameter) in a uniform, scale-independent way. A corresponding 'global' space-time then becomes such an approximation, too.

In this paper, we have shown that some problems may be easily formulated in a 'global' space-time. A typical example is the process of heat conduction described in *l*-spaces by the Eq. (3) which can easily be transformed to the 'global' Eq. (5). On the other hand, there are some problems whose 'global' formulation cannot be found in a straightforward way. As an example, the problem of heat conduction described by a hyperbolic equation has been studied. A transformation to 'global' variables x, t cannot be found by using the relations (1) and (2). As a result, if a scale-independent, global formulation expressed by Eq. (9) is accepted, i.e. if we 'enforce' a global space-time upon the process, the space-time description becomes rather strange, it yields fractal structures and so on.

There are many open questions and problems connected with the present approach. For example, would it be possible to generalize the transformation given by the relations (1) and (2) to a form

$$(x,t)\equiv F_1(\xi,\nu)$$

'mixing' space and time variables somehow? This generalization would lead to more general 'global' space-times which could be useful in the description of some thermodynamic processes and phenomena.

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References

1 J. Polchinski, Nucl. Phys. B, 231 (1984) 269.
2 G. P. Lepage, The TASI-89 Summer School, Boulder, Colorado 1989.
3 K. G. Wilson and J. Kogut, Phys. Reports, 12, 2 (1974) 75.
4 Size-Scale Effects in the Failure Mechanisms of Materials and Structures, (Ed. A. Carpinteri), E & FN SPON, London 1996.
5 M. Holeček, Physica A, 183 (1992) 236.
6 L. Nottale, Fractal Space-Time and Microphysics: Towards a Theory of Scale Relativity, World Scientific, 1993.
7 I. M. Havel, Int. J. General Systems, 24 (1996) 295.
8 D. Jou, J. Casas-Vazquez and G. Lebon, Extended Irreversible Thermodynamics, Springer–Verlag, Berlin, Heidelberg 1993.